Jacobian Conjectures: Injectivity and Dynamical Systems

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Jacobian Conjectures ask (global) questions pertaining to maps and flows based on information from their (local) Jacobian matrices. First consider the

Markus-Yamabe Conjecture (1960): Suppose \( f : \mathbb{R}^n \to \mathbb{R}^n \) is \( C^1 \) and one considers the differential equation \( x' = f(x) \) where the eigenvalues \( \lambda \) of \( f'(x) \) have \( \Re \lambda < 0 \) for all \( \lambda \) at all points \( x \) and \( f(0) = 0 \), then \( x = 0 \) is globally attracting.

Status: When \( n = 2 \), the conjecture is true (see [7], [8], and [9] for proofs). When \( n \geq 3 \) the conjecture is false; Bénar and Llibre[2] produced a four-dimensional real-analytic counter-example with a periodic orbit, while Cima et al[6] present a simple three-dimensional counter-example with divergent trajectories.

For polynomial maps satisfying the Markus-Yamabe conditions, Olech[10] made connections between global attractivity and injectivity, motivating the following conjecture:

Weak Polynomial Markus-Yamabe Conjecture: If \( f \) is a polynomial map satisfying the Markus-Yamabe conditions, then \( f \) is injective.

Status: True when \( n = 2 \) (using Olech’s work and two-dimensional proof of the Markus-Yamabe Conjecture), open for \( n \geq 3 \).

Loosening these conditions a bit leads to the

Real Jacobian Conjecture on \( \mathbb{R}^n \): Every polynomial map \( F : \mathbb{R}^n \to \mathbb{R}^n \) such that \( \det F'(x) \neq 0 \) is injective.

Status: Pinchuk[11] found a counter-example in \( n = 2 \) involving polynomials of degrees 10 and 25. This connects to the classical

(Keller) Jacobian Conjecture on \( \mathbb{R}^n \) (1939): Every polynomial map \( F : \mathbb{R}^n \to \mathbb{R}^n \) such that \( \det F'(x) \equiv 1 \) is a bijective map with a polynomial inverse.

Status: Open, even for \( n = 2 \).

This problem is usually posed over the field \( \mathbb{C}^n \). Important reductions to the Jacobian Conjecture have shown that it is sufficient to prove that all cubic-homogeneous maps are injective ([12] and [3] deal with inectivity, Bass et al.[1] and Yagzhev[13] deal with cubic-homogeneous maps). A map \( F \) is in cubic-homogeneous form if 

\[ F(x) = x - H(x) \]

where \( H(tx) = t^3 H(x) \) for all \( t \in \mathbb{R} \) and \( x \in \mathbb{R}^n \). It should be noted that the Jacobian Conjecture is false for non-polynomial maps: consider the map \((u, v) = (\sqrt{2} e^{x/2} \cos(ye^{-y}), \sqrt{2} e^{x/2} \sin(ye^{-y}))\) whose Jacobian determinant is identically one, yet it is not injective.

Note that cubic-homogeneous maps have Jacobian matrices whose eigenvalues are always one at all points. Such matrices are dubbed unipotent. A very recent study [4] has proven that all \( C^1 \) two-dimensional maps
f with unipotent Jacobians are invertible and of the form

\[ f(x, y) = (x + b\phi(ax + by) + c, \ y - \phi(ax + by) + d) \]

for some constants \(a, b, c, d\) and a \(C^1\) function \(\phi\). Similar results for higher dimensions are unknown.

These results have motivated (see [4, 5]) the following so-called

**Chamberland Conjecture:** Every \(C^1\) map \(F : \mathbb{R}^n \to \mathbb{R}^n\) such that the eigenvalues of \(F'(x)\) are uniformly bounded away from zero is injective.

**Status:** Open, even for \(n = 2\).

In light of these results and conjectures, two fundamental questions may be posed:

**Question 1:** Why are results often different in dimension two compared to higher dimensions?

**Question 2:** Why are some results dependent on the function \(f\) being a polynomial while others are not?

**References**


