SUBSPACES OF COMPUTABLE VECTOR SPACES

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Abstract. We show that the existence of a nontrivial proper subspace of a vector space of dimension greater than one (over an infinite field) is equivalent to $\text{WKL}_0$ over $\text{RCA}_0$, and that the existence of a finite-dimensional nontrivial proper subspace of such a vector space is equivalent to $\text{ACA}_0$ over $\text{RCA}_0$.

1. Introduction

This paper is a continuation of [3], which is a paper by three of the authors of the present paper. In [3], the effective content of the theory of ideals in commutative rings was studied; in particular, the following computability-theoretic results were established:

Theorem 1.1. (1) There exists a computable integral domain $R$ that is not a field such that $\deg(I) \gg 0$ for all nontrivial proper ideals $I$ of $R$.

(2) There exists a computable integral domain $R$ that is not a field such that $\deg(I) = 0'$ for all finitely generated nontrivial proper ideals $I$ of $R$.

These results immediately gave the following proof-theoretic corollaries:

Corollary 1.2. (1) Over $\text{RCA}_0$, $\text{WKL}_0$ is equivalent to the statement “Every (infinite) commutative ring with identity that is not a field has a nontrivial proper ideal.”

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(2) Over RCA\textsubscript{0}, ACA\textsubscript{0} is equivalent to the statement “Every (infinite) commutative ring with identity that is not a field has a finitely generated nontrivial proper ideal.”

In the present paper, we complement these results with related results from linear algebra. (We refer to [3] for background, motivation, and definitions.)

We start with the following

**Definition 1.3.**

(1) A computable field is a computable subset \( F \subseteq \mathbb{N} \) equipped with two computable binary operations \(+, \cdot \) on \( F \), together with two elements \( 0, 1 \in F \) such that \((F, 0, 1, +, \cdot)\) is a field.

(2) A computable vector space (over a computable field \( F \)) is a computable subset \( V \subseteq \mathbb{N} \) equipped with two computable operations \(+ : V^2 \to V\) and \( \cdot : F \times V \to V \), together with an element \( 0 \in V \) such that \((V, 0, +, \cdot)\) is a vector space over \( F \).

This notion was first studied by Dekker [2], then more systematically by Metakides and Nerode [5] and many others.

As in [3] for nontrivial proper ideals in rings, one motivation in the results below is to understand the complexity of nontrivial proper subspaces of a vector space of dimension greater than one, and the proof-theoretic axioms needed to establish their existence. For example, consider the following elementary characterization of when a vector space has dimension greater than one.

**Proposition 1.4.** A vector space \( V \) has dimension greater than one if and only if it has a nontrivial proper subspace.

As in the case of ideals in [3], we will be able to show that this equivalence is not effective, and to pin down the exact proof-theoretic strength of the statement in two versions, for the existence of a nontrivial proper subspace and of a finite-dimensional nontrivial proper subspace:

**Theorem 1.5.**

(1) There exists a computable vector space \( V \) of dimension greater than one (over an infinite computable field) such that \( \deg(W) \gg 0 \) for all nontrivial proper subspaces \( W \) of \( V \).

(2) There exists a computable vector space \( V \) of dimension greater than one (over an infinite computable field) such that \( \deg(W) \geq 0' \) for all finite-dimensional nontrivial proper subspaces \( W \) of \( V \).

Again, after a brief analysis of the induction needed to establish Theorem 1.5, we obtain the following proof-theoretic corollaries:
Corollary 1.6. (1) Over $\text{RCA}_0$, $\text{WKL}_0$ is equivalent to the statement “Every vector space of dimension greater than one (over an infinite field) has a nontrivial proper subspace.”

(2) Over $\text{RCA}_0$, $\text{ACA}_0$ is equivalent to the statement “Every vector space of dimension greater than one (over an infinite field) has a finite-dimensional nontrivial proper subspace.”

2. The proof of Theorem 1.5

For the proof of part (1) of Theorem 1.5, we begin with a few easy lemmas:

Lemma 2.1. Suppose that $V$ is a vector space, that $\{v, w\}$ is a linearly independent set of vectors in $V$, and that $u \neq 0$ is a vector in $V$. Then there exists at most one scalar $\lambda$ such that $u \in \langle v - \lambda w \rangle$.

Proof. Suppose that $u \in \langle v - \lambda_1 w \rangle$ and that $u \in \langle v - \lambda_2 w \rangle$. Fix $\mu_1, \mu_2$ such that $u = \mu_1 (v - \lambda_1 w)$ and $u = \mu_2 (v - \lambda_2 w)$. Notice that $\mu_1, \mu_2 \neq 0$ because $u \neq 0$. We now have

$$\mu_1 v - \mu_1 \lambda_1 w = u = \mu_2 v - \mu_2 \lambda_2 w,$$

and hence

$$(\mu_1 - \mu_2)v + (\mu_2 \lambda_2 - \mu_1 \lambda_1)w = 0.$$

Since $\{v, w\}$ is linearly independent, it follows that $\mu_1 - \mu_2 = 0$ and $\mu_2 \lambda_2 - \mu_1 \lambda_1 = 0$, hence $\mu_1 = \mu_2$ and $\mu_1 \lambda_1 = \mu_2 \lambda_2$. Since $\mu_1 = \mu_2 \neq 0$, it follows from the second equation that $\lambda_1 = \lambda_2$. \qed

Lemma 2.2. Suppose that $V$ is a vector space with basis $B$, which is linearly ordered by $\prec$. Suppose that

(1) $v \in V$.
(2) $e \in B$.
(3) $\lambda$ is a scalar.
(4) $e \succ \text{max}(\text{supp}(v))$ (where $\text{supp}(v) = \text{supp}_B(v)$, the support of $v$, is the finite set of basis vectors in $B$ needed to write $v$ as a linear combination in this basis).

Then $B \setminus \{e\}$ is a basis for $V$ over $\langle e - \lambda v \rangle$, and, for all $w \in V$, $\text{max}(\text{supp}_{B \setminus \{e\}}(w + \langle e - \lambda v \rangle)) \leq \text{max}(\text{supp}_B(w))$.

Proof. Notice that $e \in \langle (B \setminus \{e\}) \cup \{e - \lambda v\} \rangle$ because $e \notin \text{supp}(v)$, so $(B \setminus \{e\}) \cup \{e - \lambda v\}$ spans $V$. Suppose that $e_1, e_2, \ldots, e_n \in B \setminus \{e\}$ are distinct and $\mu_1, \mu_2, \ldots, \mu_n$ are scalars such that

$$\mu_1 e_1 + \mu_2 e_2 + \cdots + \mu_n e_n \in \langle e - \lambda v \rangle.$$
Fix \( \mu \) such that
\[
\mu_1 e_1 + \mu_2 e_2 + \cdots + \mu_n e_n = \mu(e - \lambda v)
\]
and notice that we must have \( \mu = 0 \) (by looking at the coefficient of \( e \)), hence each \( \mu_i = 0 \) because \( B \) is a basis. Therefore, \( B \setminus \{ e \} \) is a basis for \( V \) over \( \langle e - \lambda v \rangle \). By hypothesis 4, the last line of the lemma now follows easily. \( \square \)

**Lemma 2.3.** Suppose that \( V \) is a vector space with basis \( B \), which is linearly ordered by \( \prec \). Suppose that

1. \( v_1, v_2 \in V \).
2. \( e_1, e_2 \in B \) with \( e_1 \neq e_2 \).
3. \( \lambda \) is a scalar.
4. \( e_1 \succ \max(\text{supp}(v_1) \cup \text{supp}(v_2)) \).
5. \( \{v_1, e_1\} \) is linearly independent.
6. \( v_1 \notin \langle e_2 - \lambda v_2 \rangle \).

Then \( \{v_1, e_1\} \) is linearly independent over \( \langle e_2 - \lambda v_2 \rangle \).

**Proof.** Suppose that
\[
\mu_1 v_1 + \mu_2 e_1 = \mu_3 (e_2 - \lambda v_2).
\]
We need to show that \( \mu_1 = \mu_2 = 0 \).

*Case 1: \( e_1 \prec e_2 \).* In this case, we must have \( \mu_3 = 0 \) (by looking at the coefficient of \( e_2 \)). Thus, \( \mu_1 v_1 + \mu_2 e_1 = 0 \), and hence \( \mu_1 = \mu_2 = 0 \) since \( \{v_1, e_1\} \) is linearly independent.

*Case 2: \( e_1 \succ e_2 \).* In this case, we must have \( \mu_2 = 0 \) (by looking at the coefficient of \( e_1 \)). Thus, \( \mu_1 v_1 = \mu_3 (e_2 - \lambda v_2) \). Since \( v_1 \notin \langle e_2 - \lambda v_2 \rangle \), this implies that \( \mu_1 = 0 \). \( \square \)

By applying the above three lemmas in the corresponding quotient, we obtain the following results.

**Lemma 2.4.** Suppose that \( V \) is a vector space, that \( X \subseteq V \), that \( \{v, w\} \) is linearly independent over \( \langle X \rangle \), and that \( u \notin \langle X \rangle \). Then there exists at most one \( \lambda \) such that \( u \in \langle X \cup \{v - \lambda w\} \rangle \). \( \square \)

**Lemma 2.5.** Suppose that \( V \) is a vector space, that \( X \subseteq V \), and that \( B \) is a basis for \( V \) over \( \langle X \rangle \) that is linearly ordered by \( \prec \). Suppose that

1. \( v \in V \).
2. \( e \in B \).
3. \( \lambda \) is a scalar.
4. \( e \succ \max(\text{supp}(v)) \).

Then \( B \setminus \{e\} \) is a basis for \( V \) over \( \langle X \cup \{e - \lambda v\} \rangle \) and, for all \( w \in V \), \( \max(\text{supp}_{B \setminus \{e\}}(w + \langle X \cup \{e - \lambda v\} \rangle)) \leq \max(\text{supp}_B(w)) \).
Lemma 2.6. Suppose that $V$ is a vector space, that $X \subseteq V$, and that $B$ is a basis for $V$ over $\langle X \rangle$ that is linearly ordered by $\prec$. Suppose that

1. $v_1, v_2 \in V$.
2. $e_1, e_2 \in B$ with $e_1 \neq e_2$.
3. $\lambda$ is a scalar.
4. $e_1 \succ \max(\text{supp}(v_1) \cup \text{supp}(v_2))$.
5. $\{v_1, e_1\}$ is linearly independent over $\langle X \rangle$.
6. $v_1 \notin \langle X \cup \{e_2 - \lambda v_2\} \rangle$.

Then $\{v_1, e_1\}$ is linearly independent over $\langle X \cup \{e_2 - \lambda v_2\} \rangle$. \hfill \Box

Proof of Theorem 1.5. Fix two disjoint c.e. sets $A$ and $B$ such that $\deg(S) \gg 0$ for any set $S$ satisfying $A \subseteq S$ and $B \cap S = \emptyset$. Let $V^\infty$ be the vector space over the infinite computable field $F$ on the basis $e_0, e_1, e_2, \ldots$ (ordered by $\prec$ as listed) and list $V^\infty$ as $v_0, v_1, v_2, \ldots$ (viewed as being coded effectively by natural numbers). We may assume that $v_0$ is the zero vector of $V^\infty$. Fix a computable injective function $g: \mathbb{N}^3 \to \mathbb{N}$ such that $e_{g(i,j,n)} \succ \max(\text{supp}(v_i) \cup \text{supp}(v_j))$ for all $i, j, n \in \mathbb{N}$. We build a computable subspace $U$ of $V^\infty$ with the plan of taking the quotient $V = V^\infty / U$.

We have the following requirements for all $v_i, v_j \notin U$:

$$R_{i,j,n} : \text{n} \notin A \cup B \Rightarrow \text{each of } \{v_i, e_{g(i,j,n)}\} \text{ and } \{v_j, e_{g(i,j,n)}\} \text{ are linearly independent over } U,$$

$$\text{n} \in A \Rightarrow e_{g(i,j,n)} - \lambda v_i \in U \text{ for some nonzero } \lambda \in F, \text{ and }$$

$$\text{n} \in B \Rightarrow e_{g(i,j,n)} - \lambda v_j \in U \text{ for some nonzero } \lambda \in F.$$

We now effectively build a sequence $U_2, U_3, U_4, \ldots$ of finite subsets of $V$ such that $U_2 \subseteq U_3 \subseteq U_4 \subseteq \ldots$, and we set $U = \bigcup_{n \geq 2} U_n$. We also define a function $h: \mathbb{N}^4 \to \{0, 1\}$ for which $h(i, j, n, s) = 1$ if and only if we have acted for requirement $R_{i,j,n}$ at some stage $\leq s$ (as defined below). We ensure that for all $k \geq 2$, we have $v_k \in U$ if and only if $v_k \in U_k$, which will make our set $U$ computable. We begin by letting $U_2 = \{v_0\}$ and letting $h(i, j, n, s) = 0$ for all $i, j, n, s$ with $s \leq 2$.

Suppose that $s \geq 2$ and we have defined $U_s$ and $h(i, j, n, s)$ for all $i, j, n$. Suppose also that we have for any $i, j, n, s$ that $v_i, v_j \notin \langle U_s \rangle$:

1. If $h(i, j, n, s) = 0$, then each of $\{v_i, e_{g(i,j,n)}\}$ and $\{v_j, e_{g(i,j,n)}\}$ is linearly independent over $\langle U_s \rangle$.
2. If $h(i, j, n, s) = 1$ and $n \in A_s$, then $e_{g(i,j,n)} - \lambda v_i \in U_s$ for some nonzero $\lambda \in F$.
Suppose first that no such triple \( \langle i, j, n \rangle \) exists. If \( v_{s+1} \in \langle U_s \rangle \), then let \( U_{s+1} = U_s \cup \{v_{s+1}\} \); otherwise let \( U_{s+1} = U_s \). Also, let \( h(i, j, n, s+1) = h(i, j, n, s) \) for all \( i, j, n \).

Suppose then that such a triple \( \langle i, j, n \rangle \) exists, and fix the least such triple. If \( n \in A_s \), then search for the least (under some effective coding) nonzero \( \lambda \in F \) such that \( v_k \notin \langle U_s \cup \{e_{g(i,j,n)} - \lambda v_i\} \rangle \) for all \( k \leq s \) such that \( v_k \notin U_s \). (Such \( \lambda \) must exist by Lemma 2.4 and the fact that \( F \) is infinite.) Let \( U'_s = U_s \cup \{e_{g(i,j,n)} - \lambda v_i\} \) and let \( h(i, j, n, s+1) = 1 \). If \( n \in B_s \), then proceed likewise with \( v_j \) replacing \( v_i \). Now, if \( v_{s+1} \in \langle U'_s \rangle \), then let \( U_{s+1} = U'_s \cup \{v_{s+1}\} \); otherwise let \( U_{s+1} = U'_s \). Also, let \( h(i, j, n, s+1) = h(i, j, n, s) \) for all other \( i, j, n \). Using Lemma 2.6 it follows that our inductive hypothesis is maintained, so we may continue.

We can now view the quotient space \( V = V^\infty / U \) as the set of \( \langle \mathbb{N}, \leq \rangle \)-least representatives (which is a computable subset of \( V^\infty \)). Notice that \( V \) is not one-dimensional because \( \{v_1, e_{g(1,2,n)}\} \) is linearly independent over \( U \) for any \( n \notin A \cup B \) (since \( v_1, v_2 \notin U \)). Suppose that \( W \) is a nontrivial proper subspace of \( V \), and fix \( W_0 \) such that \( W = W_0 / U \). Then \( W_0 \) is a \( W \)-computable subspace of \( V^\infty \), and \( U \subset W_0 \subset V^\infty \). Fix \( v_i, v_j \in V^\infty \setminus U \) such that \( v_i \in W_0 \) and \( v_j \notin W_0 \). Let \( S = \{n : e_{g(i,j,n)} \in W_0\} \). We then have that \( S \leq_T W_0 \equiv_T W \), that \( A \subseteq S \), and that \( B \cap S = \emptyset \). Thus \( \text{deg}(S) \gg 0 \), establishing part (1) of Theorem 1.5.

Part (2) of Theorem 1.5 now follows easily from part (1) and Arslanov’s Completeness Criterion [1]: If \( W \) is a finite-dimensional nontrivial proper subspace of the above vector space \( V \) then \( W_0 \) is a c.e. set that computes a degree \( \gg 0 \); thus \( \text{deg}(W) \) must equal \( 0' \).  

3. The Proof of Corollary 1.6

As usual for these arguments, we only have to check that

(i) \( \text{WKL}_0 \) (or \( \text{ACA}_0 \), respectively) suffices to prove the existence of a (finite-dimensional) nontrivial proper subspace (establishing the left-to-right direction of Corollary 1.6); and

(ii) the above computability-theoretic arguments can be carried out in \( \text{RCA}_0 \) (establishing the right-to-left direction of Corollary 1.6).  

\[ \square \]
Part (i) just requires a bit of coding. Using $WKL_0$, one can code membership in a nontrivial proper subspace $W$ of a vector space $V$ on a binary tree $T$ where one arbitrarily fixes two linearly independent vectors $w, w' \in V$ such that $w \in W$ and $w' \notin W$ is specified. A node $\sigma \in T_W$ is now terminal if the subspace axioms for $W$ are violated along $\sigma$ using coefficients with Gödel number $< |\sigma|$, which can be checked effectively relative to the open diagram of the vector space. Using $ACA_0$, one can form the one-dimensional subspace generated by any nonzero vector in $V$.

Part (ii) boils down to checking that $\Sigma^0_1$-induction suffices for the computability-theoretic arguments from Section 2. First of all, note that the definition of $U$ and of the vector space operations on $U$ can be carried out using $\Delta^0_1$-induction. $WKL_0$ is equivalent to showing $\Sigma^0_1$-Separation, so fix any sets $A$ and $B$ that are $\Sigma^0_1$-definable in our model of arithmetic. Then their enumerations $\{A_s\}_{s \in \omega}$ and $\{B_s\}_{s \in \omega}$ exist in the model, and from them we can define the subspace $U$, the quotient space $V = V^\infty/U$, and the function mapping each vector $v \in V^\infty$ to its $\leq_N$-least representative modulo $U$, using only $\Sigma^0_1$-induction. (The latter function only requires that in $RCA_0$, any infinite $\Delta^0_1$-definable set can be enumerated in order.) The hypothesis now provides the nontrivial proper subspace $W$, and from it we can define the separating set $S$ by $\Delta^0_1$-induction.

Proving the right-to-left direction of Corollary 1.6 (2) could be done using the concept of maximal pairs of c.e. sets as in our companion paper [3]. But for vector spaces, there is actually a much simpler proof: In the above construction, simply set $A$ to be any $\Sigma^0_1$-set and $B = \emptyset$. Now $V$ must be a vector space of dimension greater than one. Since any finitely generated nontrivial proper subspace can compute a one-dimensional subspace, we may assume we are given a one-dimensional subspace $W$, spanned by $v_i$, say. But then

\[
 n \in A \text{ iff } \{v_i, e_{g(i,1,n)}\} \text{ is linearly dependent in } V \\
 n \in W \text{ iff } e_{g(i,1,n)} \in W,
\]

and so $W$ can compute $A$ as desired.

References


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