Homework 8 : Due Wednesday, April 21

Problem 1: Find (with proof) the minimal polynomial of \(\sqrt{2} - \sqrt{2}\) over \(\mathbb{Q}\).

Problem 2: Find values of \(a, b \in \mathbb{N}^+\) such that \(\left[\mathbb{Q}(\sqrt[3]{a} + \sqrt[4]{b}) : \mathbb{Q}\right] = 12\). Justify your answer.

Problem 3:
   a. Show that \(x^5 + x^2 + 1 \in (\mathbb{Z}/2\mathbb{Z})[x]\) is irreducible in \((\mathbb{Z}/2\mathbb{Z})[x]\).
   b. Show that \(3x^5 + 2x^4 - x^2 + 5\) is irreducible in \(\mathbb{Q}[x]\).

Problem 4:
   a. Show that \(\mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})\).
   b. Show that \(x^4 - 10x^2 + 1\) is irreducible in \(\mathbb{Q}[x]\).
   \textit{Hint}: It is possible to do the parts of this problem in either order.

Problem 5: Let \(p(x) = x^3 + 3x + 2\).
   a. Show that \(p(x)\) is irreducible in \(\mathbb{Q}[x]\).
   b. Show that \(p(x)\) has exactly one root in \(\mathbb{R}\).
   c. Let \(\alpha\) be the unique real root of \(p(x)\). We know from part a that \(p(x)\) is the minimal polynomial of \(\alpha\) over \(\mathbb{R}\), hence \(\{1, \alpha, \alpha^2\}\) is a basis of \(\mathbb{Q}(\alpha)\) over \(\mathbb{Q}\) by Theorem 6.14. In particular, we have \(\mathbb{Q}(\alpha) = \{a + b\alpha + c\alpha^2 : a, b, c \in \mathbb{Q}\}\) Now we clearly have \(\alpha^5 - \frac{2}{5}\alpha^3 + 42 \in \mathbb{Q}(\alpha)\). Find \(a, b, c \in \mathbb{Q}\) such that \(\alpha^5 - \frac{2}{5}\alpha^3 + 42 = a + b\alpha + c\alpha^2\)