Problem 1: Suppose that $p$ is prime, that $a, b \in \mathbb{Z}$, and that $p \nmid a$. Show that
\[
\sum_{k=0}^{p-1} \left( \frac{ak+b}{p} \right) = 0
\]

Problem 2: Suppose that $p$ is an odd prime and that $n \mid (p-1)$. Show that the set of $n^{th}$ powers forms a subgroup of $U(\mathbb{Z}/p\mathbb{Z})$ of order $\frac{p-1}{n}$.

Problem 3: Let $p$ be an odd prime.
   a. Show that a primitive root modulo $p$ must be a quadratic nonresidue modulo $p$.
   b. Show that every quadratic nonresidue modulo $p$ is a primitive root modulo $p$ if and only if $p = 2^n + 1$ for some $n \in \mathbb{N}^+$. Such primes are called Fermat primes and in fact any such prime must be of the form $2^{2^k} + 1$.

Problem 4: Consider the polynomial $f(x) = x^6 + x^4 - 4x^2 - 4$. Show that $f$ has a root modulo every prime, but $f$ has no integer roots.
   Hint: Begin by factoring $f(x) = (x^2 + 1)(x^4 - 4)$.

Problem 5: Suppose that $p$ is an odd prime.
   a. Show that if $p \equiv 1 \pmod{4}$, then the product of the quadratic residues in the set $\{1, 2, \ldots, p-1\}$ is congruent to $-1$ modulo $p$.
   b. Show that if $p \equiv 3 \pmod{4}$, then the product of the quadratic residues in the set $\{1, 2, \ldots, p-1\}$ is congruent to $1$ modulo $p$.

Problem 6: Suppose that $p > 3$ is prime. Prove that the sum of the quadratic residues in the set $\{1, 2, \ldots, p-1\}$ is congruent to $0$ modulo $p$. What is the sum equal to when $p = 3$?