Homework 24 : Due Monday, November 23

Problem 1: Given a prime $p$, let

$$\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z} \text{ and } \gcd(b, p) = 1 \right\}$$

Show that $\mathbb{Z}_{(p)}$ is a subring of $\mathbb{Q}$. $\mathbb{Z}_p$ is called the ring of integers localized at $p$.

Problem 2: Given two sets $A$ and $B$, the symmetric difference of $A$ and $B$, denoted $A \triangle B$, is

$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$

i.e. $A \triangle B$ is the set of elements in exactly one of $A$ and $B$.

Let $X$ be a set. Let $R = \mathcal{P}(X)$ be the power set of $X$, i.e. the set of all subsets of $X$. We define $+$ on $\cdot$ on elements of $R$ as follows. Given $A, B \in \mathcal{P}(X)$, let

$$A + B = A \triangle B$$

and let

$$A \cdot B = A \cap B$$

Show that with these operations, $R$ is a commutative ring with identity.

Problem 3: A Boolean ring is a ring for which $a^2 = a$ for all $a \in R$. For example, $\mathbb{Z}_2$ is a Boolean ring, as are all of the rings from Problem 2.

a. Show that if $R$ is a Boolean ring, then $a + a = 0$ for all $a \in R$.

b. Show that every Boolean ring is commutative.

c. Show that if a Boolean ring $R$ is an integral domain, then $R \cong \mathbb{Z}_2$.

Problem 4: Let $R$ be an integral domain and let $a, b \in R$. Show that $\langle a \rangle = \langle b \rangle$ if and only if there exists a unit $u \in R$ with $a = bu$.

Problem 5: Let $R = C[0, 1]$ be the ring of all continuous functions on $[0, 1]$. Let

$$I = \{ f \in C[0, 1] : f(0) = 0 = f(1) \}$$

a. Show that $I$ is an ideal of $R$.

b. Show that $I$ is not a prime ideal of $R$. 