Question (Section 5.5, #54): Evaluate \( \int_0^1 x \sqrt{1 - x^4} \, dx \) by making a substitution and interpreting the resulting integral in terms of an area.

Answer: We begin by recalling that \( x^2 + y^2 = 1 \) defines the unit circle (that is the circle of radius 1) centered at the origin. If we attempt to solve for \( y \), we first notice that \( y^2 = 1 - x^2 \), and so taking the square root we see that

\[
y = \pm \sqrt{1 - x^2}
\]

Thus, the graph of the function \( f(x) = \sqrt{1 - x^2} \) gives the top half of the unit circle, and the graph of the function \( g(x) = -\sqrt{1 - x^2} \) gives the bottom half of the unit circle.

In attempting to evaluate the integral, we begin by making the substitution \( u = x^2 \) with the hope of taking out the extra \( x \) out front. We then have that \( du = 2x \, dx \), so \( x \, dx = \frac{1}{2} \, du \). Now when \( x = 0 \), we have \( u = 0^2 = 0 \). Similarly, when \( x = 1 \), we have \( u = 1^2 = 1 \). Therefore,

\[
\int_0^1 x \sqrt{1 - x^4} \, dx = \int_0^1 \sqrt{1 - (x^2)^2} \cdot x \, dx \\
= \int_0^1 \sqrt{1 - u^2} \cdot \frac{1}{2} \, du \\
= \frac{1}{2} \int_0^1 \sqrt{1 - u^2} \, du
\]

We now examine the integral \( \int_0^1 \sqrt{1 - u^2} \, du \). Since the function \( f(u) = \sqrt{1 - u^2} \) is nonnegative on the interval \([0, 1]\), this integral is just the area of the region above the \( u \)-axis and below the graph of \( f(u) = \sqrt{1 - u^2} \) on the interval \([0, 1]\). As we noted above, the function \( f(u) = \sqrt{1 - u^2} \) on the interval \([-1, 1]\) is the graph of the top half of the unit circle. Thus, the integral

\[
\int_0^1 \sqrt{1 - u^2} \, du
\]

is calculating exactly 1/4 of the area of the unit circle. Since the unit circle has area \( \pi \), it follows that

\[
\int_0^1 \sqrt{1 - u^2} \, du = \frac{1}{4} \cdot \pi = \frac{\pi}{4}
\]

and hence

\[
\int_0^1 x \sqrt{1 - x^4} \, dx = \frac{1}{2} \int_0^1 \sqrt{1 - u^2} \, du = \frac{1}{2} \cdot \frac{\pi}{4} = \frac{\pi}{8}
\]