Calculus of Variations Assignment

1. The flight path of an aircraft is required between two points distance D apart on a level desert. The cost of flying the aircraft an infinitesimal distance ds at a height h is $\exp(-h/H)ds$ for some positive constant H (note: this says that it is cheaper to fly higher). Find the flight path which gives the minimum cost, that is, minimize the function

$$\int_0^D e^{-y/H}\sqrt{1+(y')^2}\,dx$$

What is a necessary relationship between D and H for a solution to exist?

2. A circular column of water of radius l is rotated about its vertical axis at constant angular velocity, ω, so that the surface is a radially symmetric function whose low point is at the center of the circular cross-section. Since nature is "lazy", the (upper) free surface assumes a shape which preserves the volume

$$V = 2\pi \int_0^l xy(x)dx$$

and minimizes the potential energy

$$\rho\pi \int_0^l [gy^2(x) - \omega^2 x^2 y(x)]xdx$$

where ρ is the density of the water, g is the standard gravitational constant and $y(x)$ is the height of the water at a radial distance x from the center.

   (a) Find the height function y.

   (b) What is the smallest value of ω so that the surface touches the bottom of the tank?

3. Two metal rings with a common axis are dipped into a soap solution then drawn out so that a soap film connects the two rings. The physics states that the surface tension is minimized, implying that the surface area is also minimized. If the common axis is the x-axis and one ring has a point at (a, A) and the other at (b, B), we want to minimize

$$\int_a^b 2\pi y(x)\sqrt{1+y'(x)^2}dx$$

As seen in class, the Euler-Lagrange equation (plus some DE solving tricks) leads us to claiming that any function y which minimizes the integral is a solution to the DE $y'' = ky$ for some positive constant k. If the rings have the same radius 1, what is the farthest apart the rings can be before the soap film pops?

4. Recall that Queen Dido's problem is to maximize the area of land bounded by a straight coast and a rope of fixed length L, that is, maximize

$$\int_a^b y(x)dx$$

subject to the constraints $y(x) > 0$ for $a < x < b$, $y(a) = 0$, $y(b) = 0$ and

$$\int_a^b \sqrt{1+y'(x)^2}dx = L$$

Note that the distance between a and b is not specified. How can Queen Dido maximize the area?