

Mathematical Biology

1. The following model was introduced to model the interactions between baleen whales and their main food source, krill:

$$x' = rx \left(1 - \frac{x}{K}\right) - axy$$

$$y' = sy \left(1 - \frac{y}{bx}\right)$$

The second equation indicates that the whale carrying capacity is not constant but is a function of the krill population. Do a complete phase plane analysis of this model.

2. Species may derive mutual benefit from their association: this type of interaction is known as *mutualism*. The following system describes a possible pair of mutualists:

$$x' = rx \left(1 - \frac{x}{a + by}\right)$$

$$y' = ry \left(1 - \frac{y}{c + dx}\right)$$

where $bd < 1$.

- (a) Explain why the equations describe a mutualistic interaction.
 - (b) Determine the qualitative behavior of this model by performing a phase plane analysis.
 - (c) Why is it necessary to assume that $bd < 1$?
3. When food is plentiful, a predator's appetite is soon satiated, so an increase in the prey population has little effect on the interaction terms. One system that models this growth is

$$x' = x \left(-a + \frac{by}{c + ky}\right)$$

$$y' = y \left(d - ey - \frac{fx}{c + ky}\right)$$

where a, b, \dots, k are positive constants. Consider the case where $a = 0.5$, $b = d = e = f = 1$ and $c = 0.3$. Examine the long term behavior of the system for values of k (the satiation constant) ranging from 0 to 1.4.

4. The cottony cushion scale insect had been accidentally introduced from Australia in 1868 to infest American citrus orchards. Its numbers were controlled but not eliminated by importation of the insects's natural predator, a particular kind of ladybird beetle. When DDT was first introduced as the harvesting agent, it was hoped that the scale insect could be completely wiped out. But DDT acts indiscriminantly, killing all insects it touches. This is equivalent to a so-called harvesting model:

$$x' = -x + xy/10 - Hx$$

$$y' = y - xy/5 - Hy$$

where H is the constant harvesting constant.

- (a) Examine the change in long term behavior as H varies from 0 (no harvesting) to 5 (heavy harvesting).
- (b) Determine what the average value of both x and y are as a function of H over one cycle (like we did in class for the predator-prey model). What effect did the use of DDT have on the insect populations?
5. The following model has been used in the study of infectious diseases which includes emigration of susceptibles:

$$S' = -g(I)S - \lambda S$$

$$I' = g(I)S - \gamma I$$

$$R' = \lambda S + \gamma I$$

The function $g(I)$ takes into account "psychological effects" and has the following properties: $g(0) = 0$, g is increasing up to a point, decreasing

afterwards and approaches 0 as $I \rightarrow \infty$. Explain the equations and show that the epidemic will always tend to extinction with respect to both infectives and susceptibles.