Equidissections of Kite-Shaped Quadrilaterals

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Abstract

Let $Q(a)$ be the convex kite-shaped quadrilateral with vertices $(0, 0)$, $(1, 0), (0, 1), (a, a)$, where $a > 1/2$. We wish to dissect $Q(a)$ into triangles of equal areas. What numbers of triangles are possible? Since $Q(a)$ is symmetric about the line $y = x$, $Q(a)$ admits such a dissection into any even number of triangles. In this article, we prove four results describing $Q(a)$ that can be dissected into certain odd numbers of triangles.

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1 Introduction

We wish to dissect a convex polygon K into triangles of equal areas. A dissection of K into m triangles of equal areas is called an *m-equidissection.* The *spectrum* of K, denoted $S(K)$, is the set of integers m for which K has an m-equidissection. Note that if m is in $S(K)$, then so is km for all $k > 0$. If $S(K)$ consists of precisely the positive multiples of m, we write $S(K) = \langle m\rangle$ and call $S(K)$ *principal.*

Quite a bit is known about the spectrum of the trapezoid $T(a)$ with vertices $(0, 0), (1, 0), (0, 1), (a, 1), a > 0$. For example, if a is rational, $a = r/s$ where r and s are relatively prime positive integers, then $S\big(T(a)\big) = \langle r + s\rangle$; if a is transcendental, then $S\big(T(a)\big)$ is the empty set. (See [3] or [6].) In addition, $S\big(T(a)\big)$ is known for many irrational algebraic numbers a, particularly a satisfying a quadratic polynomial. (See [1], [2], and [5].) For instance, if $a = (2r - 1) + r\sqrt{3}$ where r is an integer ≥ 8, then $S\big(T(a)\big) = \{4r, 5r, 6r, \ldots\}$.

Less is known about the spectrum of the kite-shaped quadrilateral $Q(a)$ with vertices $(0, 0), (1, 0), (0, 1), (a, a), a > 1/2$. Here certainly $S\big(Q(a)\big)$ contains 2 and hence all even positive integers. If $a = 1$, $Q(a)$ is a square, and in this case $S\big(Q(a)\big) = \langle 2\rangle$. (See [4].) For other values of a, the question is: What odd numbers, if any, are in $S\big(Q(a)\big)$? In Section 2, we prove four theorems that answer this question for certain a. In Section 3, we pose some questions that remain open.

2 Main Results

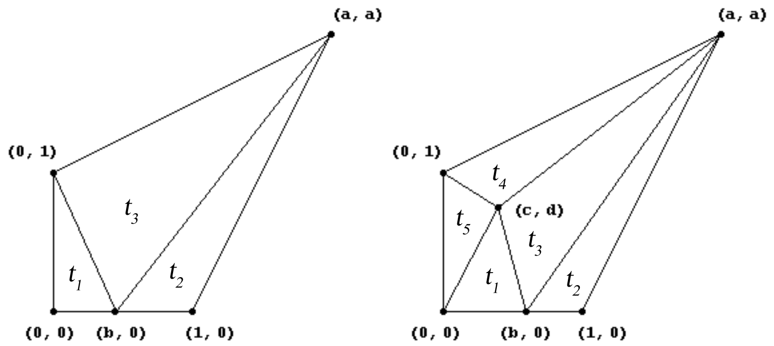
As in the introduction, $Q(a)$ denotes the quadrilateral with vertices $(0,0)$, $(1,0)$, $(0,1)$, $(a,1)$, $a > 1/2$. The following two results about $Q(a)$ are shown in [3] (pp. 290-1):

1. Let ϕ_2 be an extension to $\mathbf{R}$ of the 2-adic valuation on $\mathbf{Q}$. (See [6] for a discussion of valuations.) If $\phi_2(a) > -1$, then $S\big(Q(a)\big) = \langle 2\rangle$. In particular, if a is transcendental, then $S\big(Q(a)\big) = \langle 2\rangle$.
2. Let $a > 1/2$ be a rational number such that $\phi_2(a) \leq -1$. That is, $a = r/(2s)$, where r and s are relatively prime positive integers, r is odd, and $r > s$. Then $S\big(Q(a)\big)$ contains all odd integers of the form $r + 2sk$ for $k \geq 0$.

Two questions raised in [3] and [6] are:

- Are there rational numbers a with $\phi_2(a) \leq -1$ for which $S\big(Q(a)\big)$ contains odd numbers less than r?
- Are there irrational algebraic numbers a with $\phi_2(a) \leq -1$ for which $S\big(Q(a)\big)$ contains odd numbers? In particular, does $S\big(Q(\sqrt{3}/2)\big)$ contain odd numbers?

We answer these questions in the affirmative. First we present a slight strengthening of statement 2 above.



a) *b)*

Figure 1

<u>Theorem 1</u>: Let $a = r/(2s)$, where r and s are relatively prime positive integers, r is odd, $r > s$. Then $S\big(Q(a)\big)$ contains all integers of the form $r + 2k$ for $k \geq 0$.

Pf: Partition $Q(a)$ into three triangles as in Figure 1a). We want to find nonnegative integers t_1, t_2, t_3 so that the areas A_1, A_2, A_3 of the three triangles satisfy

$$A_1 t = at_1,\ A_2 t = at_2,\ A_3 t = at_3 \tag{1}$$

where $t = t_1 + t_2 + t_3$. (Note that the area of $Q(a)$ is a.) Then $Q(a)$ can be further dissected into t triangles each of area a/t. Here $A_1 = \frac{1}{2}b$, $A_2 = \frac{1}{2}a(1-b)$, $A_3 = \frac{1}{2}(a + ab - b)$. For $k \geq 0$, choose $t_1 = s$, $t_2 = k$, $t_3 = r - s + k$, $b = r/(r + 2k)$. Then $t = r + 2k$, $b = r/t$, and equations (1) are satisfied. Thus $r + 2k \in S\big(Q(a)\big)$. ■

<u>Theorem 2</u>: Let a be as in Theorem 1 and suppose r is not a prime number. Then $S\big(Q(a)\big)$ contains odd numbers less than r.

Pf: We know that $S\big(Q(a)\big) = S\big(Q(\frac{a}{2a-1})\big)$ for any a. (See [3], pp. 284-5.) If $a = r/(2s)$, then $a/(2a-1) = r/\big((2(r-s)\big)$. So replacing s by $r - s$ if necessary, we may assume s is odd. Partition $Q(a)$ into five triangles as shown in Figure

1b). We want the areas A_1, A_2, A_3, A_4, A_5 of the triangles to satisfy

$$A_1 t = a t_1,\ A_2 t = a t_2,\ A_3 t = a t_3,\ A_4 t = a t_4,\ A_5 t = a t_5 \tag{2}$$

where $t = t_1 + t_2 + t_3 + t_4 + t_5$. In this case, $A_1 = \frac{1}{2}bd$, $A_2 = \frac{1}{2}a(1-b)$, $A_5 = \frac{1}{2}c$, $A_4 = \frac{1}{2}\big(c(a-1) - a(d-1)\big)$, $A_3 = \frac{1}{2}\big(d(a-b) - a(c-b)\big)$. Since r is an odd composite number, we can write $r = r_1 r_2$ where $3 \leq r_1 \leq r_2$.

Case (i): $s > r_2$. Choose $t_1 = 1$, $t_2 = \frac{1}{2}(s - r_1)$, $t_3 = \frac{1}{2}(r_1 + r_2) - 1$, $t_4 = \frac{1}{2}(s - r_2)$, $t_5 = 0$, $b = r_1/s$, $c = 0$, $d = r^2/s$. Then $t = s$, and we check that equations (2) are satisfied. Then $s \in S\big(Q(a)\big)$ and $s < r$.

Case (ii): $s < r_2$. Choose $t_1 = \frac{1}{2}(r_1 - 1)$, $t_2 = \frac{1}{2}(r_1 r_2 - r_1 - 2s)$, $t_3 = \frac{1}{2}(r_2 + 1)$, $t_4 = 0$, $t_5 = \frac{1}{2}(r - r_2 - 2s)$. The assumption on s implies that the t_i are nonnegative, and their sum t is $r - 2s$. Now let $b = (t - 2t_2)/t = r_1/t$, $c = (2at_5)/t$, $d = (2at_1)/(bt) = (2at_1)/r_1$. Then $s = tt_1 - r_1 t_5$, and again we check that equations (2) are satisfied. Thus $r - 2s \in S\big(Q(a)\big)$ and $r - 2s < r$. ∎

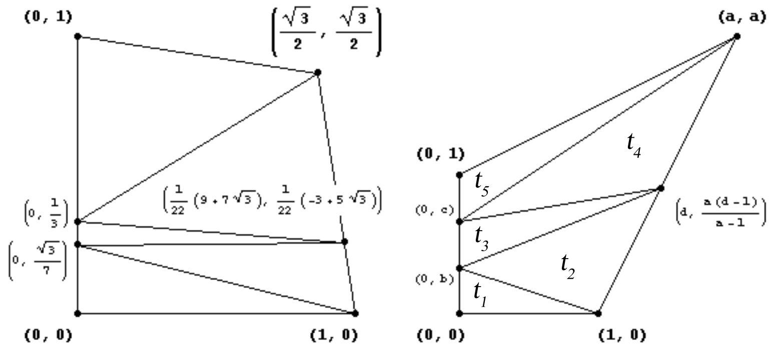
Theorem 3: Let $a = \sqrt{3}/2$. Then 21 is in $S\big(Q(a)\big)$.

Pf: Partition $Q(a)$ into five triangles shown in Figure 2a). The areas of the five triangles are in the proportion $\frac{3}{14\sqrt{3}} : \frac{3}{14\sqrt{3}} : \frac{1}{14\sqrt{3}} : \frac{7}{14\sqrt{3}} : \frac{7}{14\sqrt{3}}$ or $3 : 3 : 1 : 7 : 7$. Hence we can further dissect $Q(a)$ into $t = 3 + 3 + 1 + 7 + 7 = 21$ triangles each of area $\frac{1}{14\sqrt{3}} = \frac{1}{21}\left(\frac{\sqrt{3}}{2}\right)$. ∎

There are infinitely many radicals besides $\sqrt{3}/2$ that have odd numbers in their spectra. For example, the next theorem says $11 \in S\big(Q(\sqrt{5}/4)\big)$, $15 \in S\big(Q(\sqrt{21}/4)$, $17 \in S\big(Q(\sqrt{33}/4)$, $21 \in S\big(Q(\sqrt{65}/4)$, and so forth.

Theorem 4: For $k \geq 1$ let $a = \frac{\sqrt{(2k+1)(2k+3)}}{4\sqrt{3}}$. Then $2k + 9$ lies in $S\big(Q(a)\big)$.

Pf: Partition $Q(a)$ into five triangles as shown in Figure 2b). As before, we want the areas A_i of the triangles to satisfy equations (2) above. Here $A_1 = \frac{1}{2}b$, $A_3 = \frac{1}{2}(c - b)d$, $A_5 = \frac{1}{2}a(1 - c)$, $A_2 = \frac{1}{2}\left(\frac{d-1}{a-1}\right)(a + ab - b)$, $A_4 = \frac{1}{2}\left(\frac{a-d}{a-1}\right)(a + ac - c)$. Choose $t_1 = t_2 = t_3 = 2$, $t_5 = 3$, $t_4 = 2k$, so $t = 2k + 9$ and



a) *b)*

Figure 2

$48a^2 = (t-8)(t-6)$. Now let $b = (4a)/t$, $c = (t-6)/t$, $d = (4a)/(t-6-4a)$. We show once again that equations (2) are satisfied. Thus $2k + 9 \in S\big(Q(a)\big)$. ∎

3 Open Questions

While we have answered a few questions about odd numbers in $S\big(Q(a)\big)$, many others remain:

1. Is the converse of Theorem 2 true? That is, if a is as in Theorem 1 and r is a prime number, is r the smallest odd number in $S\big(Q(a)\big)$?
2. Let a be as in Theorem 2. What is the smallest odd number in $S\big(Q(a)\big)$? What are all the odd numbers in $S\big(Q(a)\big)$?
3. Let a be an irrational algebraic number with $\phi_2(a) \leq -1$. Does $S\big(Q(a)\big)$ always contain odd numbers?
4. Let a be arbitrary, m be an odd number. If m is in $S\big(Q(a)\big)$, is $m + 2$ in $S\big(Q(a)\big)$? (This is the same as: Is $S\big(Q(a)\big)$ closed under addition?)

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